ISRAEL JOURNAL OF MATHEMATICS 104 (1998), 301–320

A SIMPLE MAP WITH NO PRIME FACTORS

BY

ANDRÉS DEL JUNCO
Department of Mathematics, University of Toronto
Toronto, Ontario, Canada, M5S 1A1
e-mail: deljunco@math.toronto.edu

ABSTRACT

An ergodic measure-preserving transformation T of a probability space is said to be simple (of order 2) if every ergodic joining λ of T with itself is either $\mu\times\mu$ or an off-diagonal measure μ_S, i.e., $\mu_S(A\times B)=\mu(A\cap S^{-n}B)$ for some invertible, measure preserving S commuting with T. Veech proved that if T is simple then T is a group extension of any of its non-trivial factors. Here we construct an example of a weakly mixing simple T which has no prime factors. This is achieved by constructing an action of the countable Abelian group $\mathbb{Z}\oplus G$, where $G=\bigoplus_{i=1}^{\infty}\mathbb{Z}_2$, such that the $\mathbb{Z}$-subaction is simple and has centralizer coinciding with the full $\mathbb{Z}\oplus G$-action.

§1. Introduction

We consider an ergodic finite measure-preserving system (**system** for short) $\mathbf{X}=(X,\mathcal{F},\mu,T)$, that is, $(X,\mathcal{F},\mu)$ is a probability space and $T\colon X\to X$ is an invertible ergodic measure-preserving map. A (2-fold) **self-joining** of $\mathbf{X}$ is a measure λ on $\mathcal{F}\otimes\mathcal{F}$ which is $T\times T$-invariant and has both projections equal to μ. We denote by $J_e(\mathbf{X})$ or $J_e(T)$ the space of all ergodic self-joinings of $\mathbf{X}$. $C(\mathbf{X})$ will denote the centralizer of T, the set of (equivalence classes of) invertible μ-preserving maps S commuting a.e. with T. We equip $C(\mathbf{X})$ with the weak topology: $S_n\to S$ if and only if $\mu(S_n^{-1}A\triangle S^{-1}A)\to 0$. For $S\in C(\mathbf{X})$ the off-diagonal joining $\mu_S\in J_e(\mathbf{X})$, supported on the graph of S, is defined by

$$\mu_S(A)=\mu\{x\colon (x,Sx)\in A\}.$$

If $\mathbf{X}$ is weakly mixing $\mu\times\mu$ is also an ergodic self-joining.

Received February 12, 1996

$\mathbf{X}$ is said to be (2-fold) **simple** if each $\lambda \in J_e(\mathbf{X})$ is either product measure or an off-diagonal joining. $\mathbf{X}$ has **minimal self-joinings** (MSJ) if it is simple and $C(T) = \{T^n \colon n \in \mathbb{Z}\}$. Both notions can be strengthened to include higher order joinings but we will only be concerned with the two-fold versions here. We remark also that these notions have natural generalizations to actions of locally compact groups other than $\mathbb{Z}$. The definition of simplicity is due to Veech [V], who proved the following result, which was the starting point for the theory of simple maps (see [JR1], [G,H,R], [Th]).

THEOREM: *Suppose $\mathbf{X} = (X, \mathcal{F}, \mu, T)$ is simple and $\mathcal{G} \subseteq \mathcal{F}$ is a T-invariant σ-algebra. Then, modulo null sets, $\mathcal{G}$ is either the trivial σ-algebra $\{\phi, X\}$, or there is a compact subgroup K of $C(\mathbf{X})$ such that*

$$\mathcal{G} = \mathcal{I}(K) = \{A \in \mathcal{F} \colon SA = A \text{ a.e. } \forall S \in K\}.$$

In other words, $\mathbf{X}$ is a group extension of $\overline{\mathbf{X}} = (X, \mathcal{G}, \mu, T)$.

In particular, if $\mathbf{X}$ has MSJ then $\mathbf{X}$ has only the trivial factors $\{\phi, X\}$ and $\mathcal{F}$, since $C(\mathbf{X}) \simeq \mathbb{Z}$ has no non-trivial compact subgroups (see also [R]). When $\mathbf{X}$ has only the trivial factors we call $\mathbf{X}$ prime.

If $\mathbf{X}$ is simple it is not hard to see that it is either weakly mixing or has discrete spectrum [JR1]. In the second case, which may be regarded as the trivial case, every factor of $\mathbf{X}$ again has discrete spectrum, hence is not prime (unless it happens to be a cyclic permutation of prime order). On the other hand, many examples of weakly mixing simple systems are now known (although it remains an open question whether they are 'typical' in the sense of being residual in the weak topology). These include any weakly mixing group extension of a system with MSJ and the time one map of a flow with MSJ ([JR1]) as well as others ([JR2], [G,W]). All of these examples are either themselves prime or have prime factors. This leads naturally to the question, already posed in [JR1]: must a weakly mixing simple map have a non-trivial prime factor?

The purpose of this paper is to answer this question negatively. This is achieved by constructing an action of the countable discrete group $\mathbb{Z} \oplus G$, where $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$, for which the map T generating the $\mathbb{Z}$-sub-action is weakly mixing and simple, with centralizer $C(T)$ coinciding with the full $\mathbb{Z} \oplus G$ action. To see that this gives the desired example, let $G_n \simeq \bigoplus_{i=1}^{n} \mathbb{Z}_2$, $n = 1, 2, 3, \ldots$, denote the natural ascending chain of finite subgroups of $\mathbb{Z} \oplus G$, so $\mathcal{I}(G_n)$ is a descending chain of factors of T. Since every compact subgroup of $\mathbb{Z} \oplus G$ is contained in some G_n, by Veech's theorem, every factor of T contains some $\mathcal{I}(G_n)$ and hence is not prime.

The action we construct is a rank 1 action of $\mathbb{Z} \oplus G$ (the action of $\mathbb{Z}$ will not be rank one) with randomly chosen 'spacers' in the spirit of Ornstein's rank one mixing map [O] (see also [R]). What we show in effect is that almost all choices of spacers give the desired properties. The construction we make can in fact be shown to be a mixing action of $\mathbb{Z} \oplus G$, but we will not need this. With some additional conditions on the spacers, which also hold almost surely, one can show that the $\mathbb{Z}$-action is actually mixing of all orders and simple of all orders (see [JR1] for the definition of higher order simplicity). (In fact, a result of Glasner, Host and Rudolph [G,H,R] says that simplicity of order 3 implies all orders.)

There are a number of other questions about simple maps which could be answered by a generalization of the construction in this paper. Suppose, namely, that we have a locally compact group H with a closed subgroup L and an action of H for which the L-sub-action is weakly mixing and simple, with centralizer given by the action of the centralizer subgroup $C_H(L) =: \{h \in H\colon hl = lh \quad \forall l \in L\}$.

In case $H = \mathbb{Z} \times G$ and $L = \mathbb{Z}$, where G is a countable discrete group with more than one maximal finite subgroup, we obtain an example of a simple map with more than one prime factor. Such an example has already been produced in a quite different way by Glasner and Weiss [G,W], using horocycle flows. Their construction relies on the deep results of Ratner [Ra] on joinings of horocycle flows, as well as the existence of a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ with rather special properties.

In case $H = \mathbb{Z} \times G$ and $L = \mathbb{Z}$ where G is a (necessarily non-discrete) group having a compact sub-group K and $g \in G$ with $gKg^{-1} \subsetneq K$ we obtain an example of a simple map with a non-coalescent factor, answering a question posed by Mariusz Lemańczyk (see also question 3 at the end of [JLM]). Jean-Paul Thouvenot (private communication) has already produced an example using horocycle flows.

Finally, if H is a semi-direct product $\mathbb{Z}^2 \times_\alpha \mathbb{Z}$ of $\mathbb{Z}^2$ with $\mathbb{Z}$ by an automorphism α of $\mathbb{Z}^2$ of infinite order we obtain a simple action θ of $\mathbb{Z}^2$ (i.e., $\theta\colon \mathbb{Z}^2 \to \mathrm{Aut}(X,\mu)$) such that the actions θ and $\theta \circ \alpha$ are isomorphic, answering a question raised by Eli Glasner.

None of the above three examples has been worked out in detail. The first two, however, appear to be realizable using ideas similar to the present paper but the third will require substantial new ideas due to the fact that the group is highly non-abelian. One would like a construction which works under fairly general conditions on H and L and it is not yet clear what these might be. I hope to return to this question in a future paper.

We now briefly summarize the organization of this paper. In Section 2 we establish some notation and a general framework for construction of rank one group actions. We also formulate versions of the law of large numbers and the ergodic theorem which we will need. In Section 3 we construct the example. In Section 4 we establish weak mixing of the $\mathbb{Z}$-action and in Section 5 we prove that it is simple.

§2. Notation

For $m, n \in \mathbb{Z}$, $[m, n]$ will always denote the integer interval $\{m, m+1, \ldots, n\}$. μ_A denotes conditional measure on a subset A. By a **partition** P of a measurable space $(X, \mathcal{F})$ we mean a measurable map $P\colon X' \to I$ where X' is a measurable subset of X and I is a finite set, the **index set** of P. If A is a finite set **unif** A will denote the normalized counting measure on A. If $f\colon X \to Y$ is a measurable map and μ is a measure on X, $f(\mu)$ denotes $\mu \circ f^{-1}$, a measure on Y. $|A|$ denotes the cardinality of a finite set A. $A =: B$ or $B := A$ means that the symbol A is defined to mean B.

We will be dealing with many probability measures on finite sets and establishing that they are close to uniform. Closeness will always be measured in the total variation metric: if ρ and σ are probabilities on a finite set I then

$$\|\rho - \sigma\| =: \sum_{i \in I} |\rho(i) - \sigma(i)|.$$

We often write $\rho \overset{\epsilon}{\sim} \sigma$ when $\|\rho - \sigma\| < \epsilon$. Here are some easily verified properties of $\|\cdot\|$ which we will use repeatedly:

(2.1) If $A \subset I$ and $\rho(A) > 1 - \epsilon$ then $\|\rho - \rho_A\| < 2\epsilon$, if ρ_A is viewed as a measure on I.

(2.2) If $\|\rho - \sigma\| < \epsilon < \frac{\rho(A)}{2}$, $A \subset I$, then $\|\rho_A - \sigma_A\| < 3\epsilon/\rho(A)^2$.

(2.3) If $\pi\colon I \to J$ then $\|\pi(\rho) - \pi(\sigma)\| \leq \|\rho - \sigma\|$.

(2.4) Suppose μ, ν are probabilities on $I \times J$. Let $\bar{\mu}$, $\bar{\nu}$ denote the projections on I. For $x \in X$ let μ_x be the conditional measure $\mu_x(y) = \mu(x, y)/\bar{\mu}(x)$, a probability on Y. If $\mu_x = \nu_x$ $\forall x$ then $\|\mu - \nu\| = \|\bar{\mu} - \bar{\nu}\|$.

(2.5) With notation as in (2.4) if $\bar{\mu} = \bar{\nu}$ then $\|\mu - \nu\| = \sum_x \|\mu_x - \nu_x\| \bar{\mu}(x)$.

If (Y, ν) is a probability space, I is finite, $f\colon Y \to I$ is measurable and A is a measurable subset of Y we will use the notation

$$\mathrm{dist}(f(y)|y \in A) =: f(\nu_A),$$

or

$$\mathrm{dist}_\nu(f(y)|y \in A),$$

when ν is not clear from the context. Usually Y will be either a finite set, in which case ν is understood to be $\mathrm{unif}\, Y$, or Y will be a subset of a probability space (X, μ), in which case ν is understood to be μ_Y (and μ is clear from the context!). The condition $y \in A$ may be expressed by one or more statements about y, which may involve other parameters, and f may also depend on other parameters. In such cases it may be necessary to write, for example,

$$\mathrm{dist}_y(f(x,y)|g(x,y) \in B)$$

to indicate that $f(x,y)$ is to be viewed as a function of y. The justification for this somewhat imprecise notation is its flexibility. For example, the ergodic theorem says that if T is ergodic and P is a partition, then

$$\|\,\mathrm{dist}(P(T^i x)|i \in [1,n]) - \mathrm{dist}(P(x)|x \in X)\| \to 0$$

for a.a. x. Or, if $A \subset X$, then

$$\|\,\mathrm{dist}_i(P(T^i x)|1 \le i \le n, T^i x \in A) - \mathrm{dist}_x(P(x)|x \in A)\| \to 0.$$

RANK 1 CONSTRUCTIONS. We will describe a general framework for construction of rank 1 actions of a countable discrete group G. We do not assume G is Abelian, having in mind applications beyond the context of this paper. We assume G discrete for convenience but the construction could easily be generalized to locally compact G. When specialized to $G = \mathbb{Z}$ what we are describing is actually the notion of a funny rank 1 action (see [F]), that is, we do not assume that the Følner sets indexing the towers are intervals. The stronger notion of (unfunny) rank one does not generalize in an obvious way to the abstract setting.

First we describe a categorical construction of a limiting measure space. The limit will be at once inverse and direct. Suppose $A_0, A_1, A_2, \ldots$ are finite sets with subsets $B_i \subset A_i$ for $i \geq 1$ and

$$\pi_i\colon B_{i+1} \to A_i$$

are projections such that $|\pi_i^{-1}\{a\}|$ is independent of $a \in A_i$. Assume that

$$\prod_{i\geq 1} \frac{|A_i|}{|B_i|} < \infty.$$

For $i < j$ let

$$A_{i,j} = \pi_{j-1}^{-1} \cdots \pi_{i+1}^{-1} \pi_i^{-1}(A_i) \subset B_j.$$

Then we have natural projections

$$A_i = A_{i,i} \leftarrow A_{i,i+1} \leftarrow \cdots$$

so we can form the inverse limit

$$\begin{aligned} X_i &= \lim\nolimits_j A_{i,j} \\ &= \{(a_i, a_{i+1}, a_{i+2}, \ldots)\colon a_j \in A_{i,j}, \pi_j(a_{j+1}) = a_j \quad \forall j\}. \end{aligned}$$

X_i carries a unique Borel probability measure which projects onto unif $(A_{i,j})$ for all j.

We now have natural injections

$$X_0 \to X_1 \to X_2 \to \cdots$$

so we can form the direct limit

$$X = \lim\nolimits_i X_i,$$

and identify X_i with a subset of X in the natural way. Renormalizing the measures on X_i so that the injections are measure-preserving, we obtain a measure on X of total mass $\prod_i \frac{|A_i|}{|B_i|} < \infty$, and we denote by μ the corresponding normalized measure on X. Finally, we denote by $P_i\colon X_i \to A_i$ the natural projection, so P_i is a partition of $X_i \subset X$ indexed by A_i. We denote the whole set-up by

$$(X, \mathcal{F}, \mu) = \lim\nolimits_i A_i.$$

Now suppose G is a countable discrete group. The data for our construction of a rank one G-action will be two sequences $\{F_n\}$ and $\{C_n\}$ of finite subsets of G. $\{F_n\}$ should be a (not necessarily increasing) left Følner sequence in G, that is,

$$|F_n|^{-1}|gF_n\Delta F_n| \to 0$$

for each $g \in G$. C_n should be the set of centres for a partial tiling of F_{n+1} by right translation of F_n. That is, we assume that

$$F_nC_n \subset F_{n+1}$$

and, for each n and $c, c' \in C_n$ such that $c \neq c'$, we have

$$F_nc \cap F_nc' = \emptyset.$$

Finally, we assume that

$$\prod_{n\geq 1} \frac{|F_{n+1}|}{|F_nC_n|} < \infty.$$

There is a natural projection

$$\pi_n\colon F_nC_n \to F_n$$

given by

$$\pi_n(fc) = f \quad \text{for } f \in F_n, \quad c \in C_n.$$

Thus, we can form

$$(X, \mu) = \lim\nolimits_n F_n.$$

Next we make G act on X. Fixing $g \in G$, g acts in a partially defined way on F_n by left translation, namely

$$f \mapsto gf \quad \text{for } f \in F_n \cap g^{-1}F_n,$$

and these actions are consistent with the projections π_n: if $f, gf \in F_n$ and $c \in C_n$ then

$$g\pi_n(fc) = gf = \pi_n(gfc).$$

It follows that g acts partially on X_n in a well-defined way by

$$g(f_n, f_{n+1}, \ldots) = (gf_n, gf_{n+1}, \ldots).$$

The domain of definition of the action of g on X_n is $P_n^{-1}(F_n \cap g^{-1}F_n)$, so its measure goes to 1 as $n \to \infty$, since $\{F_n\}$ is Følner. Moreover, the action of g on X_{n+1} extends its action on X_n, so we obtain an a.e. defined action of g on X. Doing this $\forall g \in G$ we obtain an (a.e. defined) action of G on X, which can be viewed as left translation on the limit measure space X.

Now let us see how the standard construction of a rank one map T looks in the above language. The standard data are the heights $h_0, h_1, h_2, \ldots$ of the successive towers for T, together with the spacer heights $s_{n,1}, s_{n,2}, \ldots, s_{n,k_n}$ inserted above the nth tower before cutting and stacking. Then we can take

$$F_n = [0, h_n - 1]$$

and

$$C_n = \left\{0, h_n + s_{n,1}, 2h_n + s_{n,1} + s_{n,2}, \ldots, (k_n - 1)h_n + \sum_{i=1}^{k_n-1} s_{n,i}\right\}.$$

Note that there is a lot of freedom in the choice of $\{F_n\}$ and $\{C_n\}$. The usual construction via cutting and stacking intervals just amounts to a concrete way of realizing the limit measure space X.

We conclude this section by stating two well-known results in a form convenient for our purposes. The first of these asserts exponential convergence in the law of large numbers. It can be shown by elementary estimates using Stirling's formula (cf. [R]).

LEMMA 2.1: *Let A be a finite set, p a probability measure on A and $X_1, X_2, \ldots$ a sequence of independent A-valued random variables with common distribution p. Then $\forall \epsilon > 0 \quad \exists c = c(p,\epsilon) > 0$ and $M = M(p,\epsilon)$ such that $\forall m \geq M$*

$$P\{\|\operatorname{dist}_i(X_i | i \in [1,m]) - p\| > \epsilon\} < 2^{-cm}.$$

The next result is a version of the ergodic theorem for averages over sets of integers having a regular gap structure. It follows easily from the usual ergodic theorem.

LEMMA 2.2: *Let $(X, \mathcal{F}, \mu, T)$ be an ergodic system and $P\colon X \to A$ a partition. Given $\epsilon > 0$ and $\alpha \in (0,1)$ for a.a. $x \quad \exists N$ such that $\forall n \geq N$, $\alpha n < m < n$ and $K \in \mathbb{N}$ the following holds: Let $E = l + \bigcup_{k=0}^{K} [kn, kn+m]$ for any $l \in [-Kn, Kn]$. Then*

$$\|\operatorname{dist}_i(P(T^i x) | i \in E) - \operatorname{dist} P\| < \epsilon.$$

§3. Construction of the example

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ be the countable direct sum of copies of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, that is

$$G = \{(g_1, g_2, \ldots)\colon g_i \in \mathbb{Z}_2, g_i = 0 \text{ for all but finitely many } i\},$$

with coordinatewise addition. (We have switched to additive notation.) Let $\{e_n\}_{n=1}^{\infty}$ denote the natural sequence of generators of G and G_n the subgroup generated by $e_1, \ldots, e_n$, so $G_{n+1} = G_n \cup (e_{n+1} + G_n)$. We take G_0 to be the trivial subgroup of G.

We will construct a rank one action of $\mathbb{Z} \oplus G$ by specifying sequences $\{F_n\}$ and $\{C_n\}$ of subsets of $\mathbb{Z} \oplus G$, as in Section 2. F_n will have the form

$$F_n =: [0, h_n - 1] \times G_n,$$

for some sequence of integers $\{h_n\}$ yet to be specified. h_{n+1} will be $N_n w_n$ where $w_n = h_n + s_n$ and N_n and s_n are parameters yet to be specified. $\{F_n\}$ and $\{C_n\}$ will be defined inductively in the order

$$F_0, C_0, F_1, C_1, \ldots.$$

We define F_0 by setting $h_0 = 1$, that is F_0 is the trivial subgroup of $\mathbb{Z} \oplus G$.

Now suppose

$$F_0, C_0, F_1, C_1, \ldots, F_n$$

have been defined. We proceed to define C_n, F_{n+1} as follows. Let

$$s_n = n w_{n-1},$$
$$w_n = h_n + s_n$$

and let N_n be a parameter yet to be specified. h_{n+1} will be $N_n w_n$. Let

$$\Gamma_n = [0, N_n - 1] \times \{0, 1\},$$

and

$$S_n = [0, s_n] \times G_{n-1}.$$

C_n will have the form

$$C_n = \{(i w_n + \zeta^{\mathbb{Z}}(i, \delta), \delta e_{n+1} + \zeta^{G}(i, \delta))\colon (i, \delta) \in \Gamma_n\}$$

for an appropriate choice of the

$$\zeta(i,\delta) = (\zeta^{\mathbb{Z}}(i,\delta), \zeta^{G}(i,\delta)) \in S_n = [0, s_n] \times G_{n-1}.$$

Note that any such choice makes the $F_n c$, $c \in C_n$, disjoint and contained in F_{n+1}.

The reader should have the following picture in mind. F_{n+1} is exactly tiled by the sets $W(i,\delta) = [0, w_n - 1] \times G_n + (iw_n, \delta e_{n+1})$ (we will not use this notation in the sequel), which may be thought of as 'windows'. F_n has a 'natural' translate $F_n + (iw_n, \delta e_{n+1})$ in $W(i,\delta)$ but the translate we actually choose is the natural translate perturbed by a further translation $\zeta(i,\delta)$ which is chosen in a 'random' way (this will be made precise in a moment) and does not move $F_n + (iw_n, \delta e_{n+1})$ out of its window. The perturbation may be thought of as a 'shift' by $\zeta^{\mathbb{Z}}(i,\delta)$ in the $\mathbb{Z}$ direction and a 'rotation' by $\zeta^{G}(i,\delta)$ in the G direction. Since $\zeta^{G}(i,\delta) \in G_{n-1} \subset G_n$ this part of the perturbation does not actually move the set $F_n + (iw_n, \delta e_{n+1})$ but just permutes it within itself (hence the term 'rotation').

What we require of the $\{\zeta(i,\delta)\}_{(i,\delta)\in\Gamma_n}$ is:

(3.1) $\forall (i,\delta) \neq (i',\delta') \in \Gamma_n$ and $m \geq n^{-2} N_n$ such that $i + m \leq N_n$, $i' + m \leq N_n$ we have

$$\| \operatorname{dist}_j((\zeta(i+j,\delta), \zeta(i'+j,\delta')) | j \in [0,m]) - \operatorname{unif} S_n^2 \| \leq \frac{|S_n|^{-2}}{3n}.$$

The $\{\zeta(i,\delta)\}_{(i,\delta)\in\Gamma_n}$ will be chosen as some realization of independent random variables, also denoted $\{\zeta(i,\delta)\}_{(i,\delta)\in\Gamma_n}$, with common distribution unif (S_n). The following lemma ensures that this is possible.

LEMMA 3.1: *The probability that* (3.1) *holds goes to 1 as* $N_n \to \infty$.

Proof: Fix (i,δ), (i',δ') and m as in (3.1) and let

$$Z_j = (\zeta(i+j,\delta), \zeta(i'+j,\delta')$$

for $j \in [0,m]$. Suppose first that $\delta \neq \delta'$. Then $\{Z_j\}_{j\in[0,m]}$ is a sequence of independent random variables with distribution unif (S_n^2). It follows from Lemma 2.1 that there is a constant c such that (3.1) fails (for this particular (i,δ), (i',δ'), m) with probability less than

$$2^{-cm} \leq 2^{-cn^{-2}N_n},$$

provided N_n, and hence m, is sufficiently large.

If $\delta = \delta'$ and $i \neq i'$, we can still obtain the same conclusion, even though the Z_j are no longer independent. Simply observe that $[0, m]$ can be divided into subsets A and B, both of cardinality at least $(m-1)/2$ such that the doubletons $\{i+j, i'+j\}$, $j \in A$, are pairwise disjoint, and the same for the $\{i+j, i'+j\}$, $j \in B$. It follows that for $C = A$ or B the $\{Z_j\}_{j \in C}$ are independent, so by Lemma 2.1 we have that

$$\|(\operatorname{dist} Z_j | j \in C) - \operatorname{unif}(S_n^2)\| \leq n^{-1} \tag{3.2}$$

fails with probability less than

$$2^{-c|C|} \leq 2^{-\frac{c(m-1)}{2}} \leq 2^{-\frac{c(n^{-2}N_n - 1)}{2}}$$
$$\leq 2^{-c_1 n^{-2} N_n},$$

provided N_n, and hence $|C|$, is sufficiently large ($c_1 < c$ denotes another positive constant). If (3.1) fails (for our (i, δ), (i', δ'), m) (3.2) must fail for A or B, so we conclude that (3.1) fails with probability less than

$$2 \cdot 2^{-c_1 n^{-2} N_n} \leq 2^{-c_2 n^{-2} N_n} \qquad (0 < c_2 < c_1).$$

Since there are fewer than $4N_n^3$ choices for $((i, \delta), (i', \delta'), m)$, (3.1) holds for all (i, δ), (i', δ'), m with probability greater than

$$1 - 4N_n^3 2^{-c_2 n^{-2} N_n},$$

which approaches 1 as $N_n \to \infty$ (n is fixed!). ∎

We now assume that N_n and $\{\zeta(i, \delta)\}_{(i,\delta) \in \Gamma_n}$ have been chosen so that (3.1) holds. In addition we assume that N_n grows exponentially with n, which we may do since each N_n can be chosen as large as we please.This defines C_n and F_{n+1}, so our inductive construction of the sequences $\{F_n\}$ and $\{C_n\}$ is complete. To check finiteness of the resulting rank one construction we need to see that

$$\prod_n \frac{|F_{n+1}|}{|F_n + C_n|} < \infty.$$

Since

$$\frac{|F_{n+1}|}{|F_n + C_n|} = \frac{w_n}{h_n} = 1 + \frac{n w_{n-1}}{h_n}$$
$$= 1 + \frac{n}{N_{n-1}},$$

and N_n grows exponentially, finiteness follows.

For future use we observe that (3.1) implies, via (2.2), that $\forall(i,\delta),(i',\delta')$, m as in (3.1)

(3.3) $\sup_{\zeta_0\in S_n}\|\operatorname{dist}_j(\zeta(i+j,\delta)|0\le j\le m,\zeta(i'+j,\delta')=\zeta_0)-\operatorname{unif}(S_n)\|<n^{-1}$.

As described in Section 2, the $\{F_n\}$ and $\{C_n\}$ give rise to a rank one $\mathbb{Z}\oplus G$ action on $(X,\mu)=\lim_n F_n$. We refer to the translates F_n+c, $c\in C_n$, as n-**blocks** in F_{n+1}. The natural partition $P_n\colon X_n\to F_n$ will be called the n-**block partition**. We have

$$P_n(x)=(P_n^{\mathbb{Z}}(x),P_n^G(x))\in[0,h_n-1]\times G_n$$

and we refer to $P_n^{\mathbb{Z}}(x)$ and $P_n^G(x)$ as the $\mathbb{Z}$- and G-n-block partitions respectively. We also define the n-window partition $W_n\colon X_{n+1}\to[0,w_n-1]\times G_n$ by

$$\begin{aligned}W_n(x)&=(W_n^{\mathbb{Z}}(x),W_n^G(x))\\&=(P_{n+1}^{\mathbb{Z}}(x)\bmod w_n,P_{n+1}^G(x)\bmod e_{n+1})\end{aligned}$$

and the n-order partition $O_n\colon X_{n+1}\to\Gamma_n=[0,N_n-1]\times\{0,1\}$ by

$$O_n(x)=(O_n^{\mathbb{Z}}(x),O_n^G(x))$$

where

$$\begin{aligned}P_{n+1}^{\mathbb{Z}}(x)&=O_n^{\mathbb{Z}}(x)w_n+W_n^{\mathbb{Z}}(x),\\P_{n+1}^G(x)&=O_n^G(x)\cdot e_{n+1}+W_n^G(x).\end{aligned}$$

The definitions of O_n and W_n imply that

$$\begin{aligned}P_{n+1}(x)&=O_n(x)\cdot(w_n,e_{n+1})+W_n(x)\\&=O_n(x)\cdot(w_n,e_{n+1})+\zeta(O_n(x))+W_n(x)-\zeta(O_n(x)).\end{aligned}$$

Since $O_n(x)\cdot(w_n,e_{n+1})+\zeta(O_n(x))\in C_n$ we get

$$P_n(x)=W_n(x)-\zeta(O_n(x)) \tag{3.4}$$

provided $W_n(x)-\zeta(O_n(x))\in F_n$. This is certainly the case if $W_n^{\mathbb{Z}}(x)=j$ with

$$s_n\le j\le w_n-s_n.$$

Let us call such j **interior** (in $[0,w_n-1]$). So, we know that (3.4) holds whenever $j=W_n^{\mathbb{Z}}(x)$ is interior.

§4. Weak mixing of the $\mathbb{Z}$-action

We let T denote the action of $(1,0) \in \mathbb{Z} \oplus G$ acting on X. T can in fact be shown to be mixing, but we need only weak mixing.

PROPOSITION 4.1: *T is weakly mixing.*

Proof: We will show that the sequence $w_n = h_n + s_n$ is mixing for T, that is

$$\mu(T^{-w_n} A \cap B) \to \mu(A)\mu(B) \qquad \forall A, B \in \mathcal{F},$$

which implies weak mixing. Since the partitions P_k generate $\mathcal{F}$, to show w_n mixing it suffices to show that for each k

$$\| \operatorname{dist}_x(P_k(T^{w_n}(x)), P_k(x)) - \operatorname{unif}(F_k^2) \| \to 0.$$

Since P_{n-1} refines P_k for large n the above convergence will follow, in view of 2.3, from

$$\| \operatorname{dist}_x(P_{n-1}(T^{w_n} x), P_{n-1}(x)) - \operatorname{unif} F_{n-1}^2 \| \to 0.$$

In view of (2.5) this will in turn follow from

$$\text{(4.1)} \qquad \sup_{f \in F_n} \| \operatorname{dist}_x(P_{n-1}(T^{w_n} x) | P_{n-1}(x) = f) - \operatorname{unif} F_{n-1} \| \to 0,$$

which is what we will actually show.

Fix $f = (j, g) \in [0, w_n - 1] \times G_n$ with j interior in $[0, w_n - 1]$ and $\zeta \in S_n$ and let

$$E_{f,\zeta} =: \bigcup \{ O_n^{-1}(i,\delta) \cap W_n^{-1}(f) \colon \zeta_n(i,\delta) = \zeta,\ 0 \le i \le N_n - 2,\ \delta = 0 \text{ or } 1 \}$$
$$\subset X_{n+1} \cap T^{-w_n} X_{n+1},$$

a union of atoms of P_{n+1}. If A is an atom of P_{n+1} with

$$A \subset O_n^{-1}(i,\delta) \cap W_n^{-1}(f),$$

then

$$T^{w_n} A \subset O_n^{-1}(i+1,\delta) \cap W_n^{-1}(f)$$
$$\subset P_n^{-1}(f - \zeta(i+1,\delta)),$$

by (3.4), since j is interior. It follows that

$$\rho =: \operatorname{dist}_x(P_n(T^{w_n} x) | x \in E_{f,\zeta}) = \frac{1}{2}(\rho_0 + \rho_1)$$

where, for $\delta = 0, 1$,

$$\rho_\delta =: \text{dist}_i(f - \zeta(i+1,\delta)|\zeta(i,\delta) = \zeta, i \in [0, N_n - 2]).$$

Now, by (3.3),

$$\rho_\delta \overset{n^{-1}}{\sim} \text{unif}\,(f - S_n)$$

(taking $i = 1, \delta = 0$ or 1, $m = N_n - 2$ in (3.3)). Thus we also have

$$\rho \overset{n^{-1}}{\sim} \text{unif}\,((j,g) - S_n).$$

Let us denote $(j,g) - S_n$ by Σ. Note that Σ has the form

$$(j - [0, w_n - 1]) \times G_{n-1} \quad \text{or} \quad (j - [0, w_n - 1]) \times (e_n + G_{n-1}).$$

Since $s_n = nw_{n-1}$, clearly Σ contains at least $n-1$ full $n-1$-blocks in F_n, that is $\exists C' \subset u.c._{n-1}$, such that $|C'| \geq n - 1$ and

$$\Sigma \supset F_{n-1} + C' := \Sigma'.$$

Note that

$$|\Sigma'|/|\Sigma| \geq \frac{(n-1)h_{n-1}}{nw_{n-1}} = \alpha_n \to 1$$

as $n \to \infty$. Denoting $\Sigma'' = \Sigma \cap (F_{n-1} + C_{n-1})$ we have

$$\Sigma \supset \Sigma'' \supset \Sigma'$$

and

$$\text{dist}(P_{n-1}(T^{w_n}x)|x \in E_{f,\zeta}) = \pi_{n-1}\rho_{\Sigma''},$$

where $\pi_{n-1}\colon F_{n-1} + C_{n-1} \to F_{n-1}$ is the canonical projection. Now $\rho \overset{n^{-1}}{\sim} \text{unif}\,\Sigma$ and $|\Sigma''| \geq \alpha_n|\Sigma|$ imply, via (2.2), that

$$\rho_{\Sigma''} \overset{4n^{-1}}{\sim} \text{unif}\,\Sigma''$$

for large n. Moreover (2.1) implies

$$\text{unif}\,\Sigma'' \overset{2(1-\alpha_n)}{\sim} \text{unif}\,\Sigma'.$$

Putting this all together, we get

$$\begin{aligned}\pi_{n-1}\rho_{\Sigma''} &\overset{4n^{-1}}{\sim} \pi_{n-1}(\,\mathrm{unif}\,\Sigma'') \\ &\overset{2(1-\alpha_n)}{\sim} \pi_{n-1}(\,\mathrm{unif}\,\Sigma') \\ &= \mathrm{unif}\,F_{n-1}.\end{aligned}$$

We have shown that

$$\mathrm{dist}(P_{n-1}(T^{w_n}x)|x \in E_{f,\zeta}) \overset{\delta_1(n)}{\sim} \mathrm{unif}\,F_{n-1}, \tag{4.2}$$

where $\delta_1(n) = 4n^{-1} + 2(1-\alpha_n) \to 0$ as $n \to \infty$.

Now let us call $j_0 \in [0, h_n - 1]$ interior if $s_n \le j_0 \le h_n - ws_n$. If $f_0 = (j_0, g_0) \in F_n$ and j_0 is interior then

$$P_n^{-1}(f_0) = \cup\{E_{f,\zeta}\colon f - \zeta = f_0\}$$

and all the $f = (j, g)$ appearing in the union have j interior in $[0, w_n - 1]$. Thus by (4.2)

$$\mathrm{dist}(P_{n-1}(T^{w_n}x)|x \in P_n^{-1}(f_0)) \overset{\delta_1(n)}{\sim} \mathrm{unif}\,F_{n-1}. \tag{4.3}$$

Finally, each atom $P_{n-1}^{-1}(f)$ of P_{n-1} is a union of atoms $P_n^{-1}(f_0)$, $f_0 = (j_0, g_0)$, of which all but a fraction less than

$$\delta_2(n) =: \frac{2s_n}{h_n} + \frac{2}{N_n - 1} \to 0 \quad \text{as } n \to \infty$$

have j_0 interior, so (4.3) implies

$$\mathrm{dist}_x(P_{n-1}(T^{w_n}x)|P_{n-1}(x) = f) \overset{\delta_3(n)}{\sim} \mathrm{unif}\,F_{n-1},$$

where $\delta_3(n) =: \delta_1(n) + 2\delta_2(n) \to 0$ as $n \to \infty$. This establishes (4.1) and completes the proof of Theorem 4.1. ∎

§5. Simplicity

We denote by $T^{(n,g)}$ the action of $(n, g) \in \mathbb{Z} \oplus G$ on X, so $T = T^{(1,0)}$. The aim of this section is to prove the following.

THEOREM 5.1: *If $\lambda \in J_e(T)$ and $\lambda \neq \mu \times \mu$ then $\exists (n,g) \in \mathbb{Z} \oplus G$ such that λ is the off-diagonal measure $\mu_{T^{(n,g)}}$. In particular*

$$C(T) = \{T^{(n,g)}\colon (n,g) \in \mathbb{Z} \oplus G\}.$$

To begin the proof we introduce some terminology. For $x \in X$, by a $\mathbb{Z}-n$-**block** in x we mean any interval $[k, k+h_n-1]$ in $\mathbb{Z}$ such that $P_n^{\mathbb{Z}}(T^k x) = 0$. By a $\mathbb{Z}-n$-**window** in x we mean any $[k, k+w_n-1]$ such that $W_n^{\mathbb{Z}}(T^k x) = 0$. Each $\mathbb{Z}-n+1$-block in x is a union of N_n adjacent $\mathbb{Z}-n$-windows in x, each of which contains a single $\mathbb{Z}-n$-block. By the **order** of a $\mathbb{Z}-n$-window I in x we mean the constant value of $W_n(T^i x)$ for $i \in I$. The **time** 0 $\mathbb{Z}-n$**-window** in x, denoted $I_n(x)$, is the $\mathbb{Z}-n$-window in x containing $0 \in \mathbb{Z}$. It is defined for $x \in X_{n+1}$, hence for all sufficiently large n.

Let X^* consist of those $x \in X$ such that, for all sufficiently large n, $I_n(x)$ is not among the first or last $n^{-2}N_n$ n-windows in $I_{n+1}(x)$, so $\mu(X^*) = 1$ by the Borel–Cantelli lemma. Since λ is an ergodic joining we may now choose and fix once and for all $x, x' \in X^*$ such that (x, x') satisfies the conclusion of Lemma 2.2 applied to $(X \times X, \mathcal{F} \otimes \mathcal{F}, \lambda, T \times T)$ for $\alpha = \frac{1}{4}$, all $\epsilon > 0$ and all partitions $P_n \times P_n, n = 1, 2, \ldots$. Henceforth we denote $I_n(x)$ simply by I_n. We note that since λ is a joining, x and x' individually satisfy the corresponding conditions for the partitions P_k in the system $(X, \mathcal{F}, \mu, T)$. We will refer to these conditions on x, x' and (x, x') as genericity.

Let us call n an **offset time** if there is a $\mathbb{Z}-n$-window J in x' such that $|I_n \cap J| > \frac{1}{4}|I_n|$ and the orders of I_n and J differ. Of course, there are just two candidates for J in x'. Note that the orders of $\mathbb{Z}-n$-windows have both a $\mathbb{Z}$- and a G-component, so it suffices that one of these differs.

LEMMA 5.2: *If there are only finitely many offset times, then x, x' lie in the same $\mathbb{Z} \oplus G$-orbit, so $\lambda = \mu_{T^{(n,g)}}$ for some $(n,g) \in \mathbb{Z} \oplus G$.*

Proof: First note that, because $x, x' \in X^*$ and $n^{-2}N_n$ grows exponentially , for sufficiently large n there are plenty of $\mathbb{Z}-n$-windows in $I_{n+1}(x)$ and $I_{n+1}(x')$ respectively, both to the right and the left of $0 \in \mathbb{Z}$. If n is not an offset time, look at the two adjacent $\mathbb{Z}-n$-windows in x' which intersect I_n, both of which are contained in $I_{n+1}(x')$. Since these have different orders, one of them, call it J_n, must have the same order as I_n and satisfy

$$|I_n \cap J_n| \geq \frac{3}{4} w_n.$$

Since there are only finitely many offset times, J_n is defined for all n greater than some N. Note that J_n need not be $I_n(x')$.

Now by shifting x and x' by some T^k, $|k| \le \frac{1}{4}h_N$ we may assume that $J_N = I_N(x')$. Since I_N and J_N overlap and have the same order we conclude that

$$\begin{aligned}|I_{N+1} \cap I_{N+1}(x')| &\ge w_{N+1} - w_N - s_N \\ &\ge \frac{3}{4} w_{N+1},\end{aligned}$$

if we assume, as we may, that N is sufficiently large. This means that $I_{N+1}(x') = J_{N+1}$, and proceeding inductively, we get

$$I_n(x') = J_n \qquad \forall n \ge N.$$

Thus our assumption that I_n and J_n have the same order simply becomes

$$O_n(x) = O_n(x') \qquad \forall n \ge N.$$

It is then an easy exercise to show that this forces x and x' to be in the same $\mathbb{Z} \oplus G$ -orbit. ∎

The remainder of this section is devoted to showing that if there are infinitely many offset times then λ must be $\mu \times \mu$. This we do by an argument similar in flavour to the proof of weak mixing in the previous section.

For each offset n we now let J_n denote a $\mathbb{Z} - n$-window in x', whose order disagrees with that of I_n, such that

$$|I_n \cap J_n| \ge \frac{1}{4} w_n.$$

Let $K_n = I_n \cap J_n$ and

$$E_n = \bigcup \{K_n + i w_n \colon 0 \le i \le n^{-2} N_n\}.$$

Since $x, x' \in X^*$ we have that E_n is contained in both the $\mathbb{Z} - n + 1$-blocks in x and x' containing I_n and J_n respectively.

By our choice of (x, x') we have

$$\text{dist}_l((P_k(T^l x), P_k(T^l x'))| l \in E_n) \to \text{dist}_\lambda(P_k(x), P_k(x')) \quad \text{as } n \to \infty,$$

for all $k > 0$, so it suffices to show that

$$\text{dist}_l((P_k(T^l x), P_k(T^l x'))| l \in E_n) \to \text{ unif } (F_k^2).$$

Since P_{n+1} refines P_k, it suffices to show that

$$\|\operatorname{dist}_l((P_{n-1}(T^l x), P_{n-1}(T^l x'))|l \in E_n) - \operatorname{unif}(F_{n-1}^2)\| \to 0 \quad \text{as } n \to \infty.$$

This in turn will be implied via (2.4), (2.5) and genericity of x', by

$$(5.1) \quad \sup_{f \in F_{n-1}} \|\operatorname{dist}(P_{n-1}(T^l x)|l \in E_n, P_{n-1}(T^l x') = f) - \operatorname{unif}(F_{n-1})\| \to 0,$$

which is what we will actually show.

Now let us fix an offset time n. Let (i_0, δ_0) and (i_0', δ_0') be the orders of I_n and J_n respectively. As in the proof of Theorem 4.1, let us call j_0 interior in $[0, w_n - 1]$ if $s_n \leq j_0 \leq w_n - s_n$. Similarly, we will say j is interior in K_n if j is not among the first or last s_n integers in K_n. Now fix j interior in K_n, so $W_n(T^j x) = (j_0, g_0)$ with j_0 interior in $[0, w_n - 1]$. Note that

$$W_n^G(T^l x)(x) = g_0 \qquad \forall l \in E_n.$$

Moreover, for $0 \leq i \leq n^{-2} N_n$

$$W_n(T^{j+iw_n} x) = (j_0, g_0),$$
$$O_n(T^{j+iw_n} x) = (i_0 + i, \delta_0)$$

and

$$O_n(T^{j+iw_n} x') = (i_0' + i, \delta_0').$$

Thus for $0 \leq i \leq n^{-2} N_n$

$$\begin{aligned} P_n(T^{j+iw_n} x) &= W_n(T^{j+iw_n} x) - \zeta(O_n(T^{j+iw_n} x)) \\ &= (j_0, g_0) - \zeta(i_0 + i, \delta_0). \end{aligned}$$

Next, fix $\zeta \in S_n$ and let

$$\begin{aligned} E_{j,\zeta} &= \{j + iw_n \colon 0 \leq i \leq n^{-2} N_n, \zeta(O_n(T^{j+iw_n} x')) = \zeta\} \\ &= \{j + iw_n \colon 0 \leq i \leq n^{-2} N_n, \zeta(i_0' + i, \delta_0') = \zeta\}. \end{aligned}$$

From the above remarks we see that

$$\begin{aligned} &\operatorname{dist}_l(P_n(T^l x)|l \in E_{j,\zeta}) \\ &= \operatorname{dist}_i(P_n(T^{j+iw_n} x)|0 \leq i \leq n^{-2} N_n, \zeta(i_0' + i, \delta_0') = \zeta) \\ &= \operatorname{dist}_i((j_0, g_0) - \zeta(i_0 + i, \delta_0)|0 \leq i \leq n^{-2} N_n, \zeta(i_0' + i, \delta_0') = \zeta) \\ &\overset{n^{-1}}{\sim} \operatorname{unif}((j_0, g_0) - S_n), \end{aligned}$$

by (3.3), since $(i_0, \delta_0) \neq (i'_0, \delta'_0)$. As we saw in the proof of Theorem 4.1, this last distribution projects to a distribution on F_{n-1} which is close to uniform. So we have

$$\text{(5.2)} \qquad \|\operatorname{dist}_l(P_{n-1}(T^l x) | l \in E_{j,\zeta}) - \operatorname{unif}(F_{n-1})\| \leq \delta_1(n),$$

where $\delta_1(n) \to 0$ as $n \to \infty$.

Evidently there are constants $c \in \mathbb{Z}$ and $g'_0 \in G_n$ such that

$$\forall j \in K_n, \forall i \in [0, n^{-2}N_n], \qquad W_n(T^{j+iw_n}x') = (j + c, g'_0).$$

So, if $f_0 = (j_0, g_0) \in F_n$, with j_0 interior in $[0, h_n - 1]$, then

$$\{l \in E_n \colon P_n(T^l x') = f\} = \bigcup \{E_{j,\zeta} \colon (j + c, g'_0) - \zeta = f\},$$

and all the j's appearing in the union on the right are interior in K_n. Thus (5.2) implies that

$$\text{(5.3)} \qquad \|\operatorname{dist}_l(P_{n-1}(T^l x) | l \in E_n, P_n(T^l x') = f) - \operatorname{unif} F_{n-1}\| < \delta_1(n).$$

Finally, for $f = (j, g) \in F_{n-1}$, the atom $P_{n-1}^{-1}(f)$ of P_{n-1} is a union of atoms $P_n^{-1}(f_0), f_0 = (j_0, g_0)$, of which all but a fraction less than

$$\delta_2(n) = \frac{2s(n)}{\frac{1}{4}h_n} + \frac{2}{\frac{1}{4}N_n} \to 0 \quad \text{as } n \to \infty$$

have j_0 interior in $[0, h_n - 1]$. Thus (5.3) implies

$$\operatorname{dist}_l(P_{n-1}(T^l x) | l \in E_n, P_{n-1}(T^l x') = f) \overset{\delta_3(n)}{\sim} \operatorname{unif} F_{n-1},$$

where $\delta_3(n) = \delta_1(n) + 2\delta_2(n) \to 0$ as $n \to \infty$. This establishes (5.1), concluding the proof of Theorem 5.1. ∎

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